NIKULIN INVOLUTIONS ON K3 SURFACES

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Abstract. We study the maps induced on cohomology by a Nikulin (i.e. a symplectic) involution on a K3 surface. We parametrize the eleven dimensional irreducible components of the moduli space of algebraic K3 surfaces with a Nikulin involution and we give examples of the general K3 surface in various components. We conclude with some remarks on Morrison-Nikulin involutions, these are Nikulin involutions which interchange two copies of $E_8(-1)$ in the Néron Severi group.

In his paper [Ni1] Nikulin started the study of finite groups of automorphisms on K3 surfaces, in particular those leaving the holomorphic two form invariant, these are called *symplectic*. He proves that when the group G is cyclic and acts symplectically, then $G \cong \mathbf{Z}/n\mathbf{Z}$, $1 \leq n \leq 8$. Symplectic automorphisms of K3 surfaces of orders three, five and seven are investigated in the paper [GS]. Here we consider the case of $G \cong \mathbf{Z}/2\mathbf{Z}$, generated by a symplectic involution ι. Such involutions are called *Nikulin involutions* (cf.[Mo, Definition 5.1]). A Nikulin involution on the K3 surface X has eight fixed points, hence the quotient $\bar{Y} = X/\iota$ has eight nodes, by blowing them up one obtains a K3 surface Y.
In the paper [Mo] Morrison studies such involutions on algebraic K3 surfaces with Picard number $\rho \geq 17$ and in particular on those surfaces whose Néron Severi group contains two copies of $E_8(-1)$. These K3 surfaces always admit a Nikulin involution which interchanges the two copies of $E_8(-1)$. We call such involutions *Morrison-Nikulin involutions*.
The paper of Morrison motivated us to investigate Nikulin involutions in general. After a study of the maps on the cohomology induced by the quotient map, in the second section we show that an algebraic K3 surface with a Nikulin involution has $\rho \geq 9$ and that the Néron Severi group contains a primitive sublattice isomorphic with $E_8(-2)$. Moreover if $\rho = 9$ (the minimal possible) then the following two propositions are the central results in the paper:

Proposition 2.2. Let X be a K3 surface with a Nikulin involution ι and assume that the Néron Severi group $NS(X)$ of X has rank nine. Let L be a generator of $E_8(-2)^\perp \subset NS(X)$ with $L^2 = 2d > 0$ and let

$$\Lambda_{2d} := \mathbf{Z}L \oplus E_8(-2) \quad (\subset NS(X)).$$

Then we may assume that L is ample and:

(1) in case $L^2 \equiv 2 \bmod 4$ we have $\Lambda_{2d} = NS(X)$;

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(2) in case $L^2 \equiv 0 \bmod 4$ we have that either $NS(X) \cong \Lambda_{2d}$ or $NS(X) \cong \Lambda_{\widetilde{2d}}$ where $\Lambda_{\widetilde{2d}}$ is the unique even lattice containing Λ_{2d} with $\Lambda_{\widetilde{2d}}/\Lambda_{2d} \cong \mathbf{Z}/2\mathbf{Z}$ and such that $E_8(-2)$ is a primitive sublattice of $\Lambda_{\widetilde{2d}}$.

Proposition 2.3. Let $\Gamma = \Lambda_{2d}$, $d \in \mathbf{Z}_{>0}$ or $\Gamma = \Lambda_{\widetilde{2d}}$, $d \in 2\mathbf{Z}_{>0}$. Then there exists a K3 surface X with a Nikulin involution ι such that $NS(X) \cong \Gamma$ and $(H^2(X,\mathbf{Z})^\iota)^\perp \cong E_8(-2)$.

The coarse moduli space of Γ-polarized K3 surfaces has dimension 11 and will be denoted by $\mathcal{M}_{2d}$ if $\Gamma = \Lambda_{2d}$ and by $\mathcal{M}_{\widetilde{2d}}$ if $\Gamma = \Lambda_{\widetilde{2d}}$.

Thus we classified all the algebraic K3 surfaces with Picard number nine with a Nikulin involution. For the proofs we use lattice theory and the surjectivity of the period map for K3 surfaces. We also study the ι^*-invariant line bundle L on the general member of each family, for example in Proposition 2.7 we decompose the space $\mathbf{P}H^0(X,L)^*$ into ι^*-eigenspaces. This result is fundamental for the description of the ι-equivariant map $X \longrightarrow \mathbf{P}H^0(X,L)^*$. In section three we discuss various examples of the general K3 surface in these moduli spaces, recovering well-known classical geometry in a few cases. We also describe the quotient surface $\bar{Y}$.

In the last section we give examples of K3 surfaces with an elliptic fibration and a Nikulin involution which is induced by translation by a section of order two in the Mordell-Weil group of the fibration. Such a family has only ten moduli, and the minimal resolution of the quotient K3 surface Y is again a member of the same family. By using elliptic fibrations we also give an example of K3 surfaces with a Morrison-Nikulin involution. These surfaces with involution are parametrized by three dimensional moduli spaces. The Morrison-Nikulin involutions have interesting applications towards the Hodge conjecture for products of K3 surfaces (cf. [Mo], [GL]). In section 2.4 we briefly discuss possible applications of the more general Nikulin involutions.

1. General results on Nikulin Involutions

1.1. **Nikulin's uniqueness result.** A Nikulin involution ι of a K3 surface X is an automorphism of order two such that $\iota^*\omega = \omega$ for all $\omega \in H^{2,0}(X)$. That is, ι preserves the holomorphic two form and thus it is a symplectic involution. Nikulin, [Ni1, Theorem 4.7], proved that any abelian group G which acts symplectically on a K3 surface, has a unique, up to isometry, action on $H^2(X,\mathbf{Z})$.

1.2. **Action on cohomology.** D. Morrison ([Mo, proof of Theorem 5.7],) observed that there exist K3 surfaces with a Nikulin involution which acts in the following way on the second cohomology group:

$$\iota^* : H^2(X,\mathbf{Z}) \cong U^3 \oplus E_8(-1) \oplus E_8(-1) \longrightarrow H^2(X,\mathbf{Z}), \qquad (u,x,y) \longmapsto (u,y,x).$$

Thus for *any* K3 surface X with a Nikulin involution ι there is an isomorphism $H^2(X,\mathbf{Z}) \cong U^3 \oplus E_8(-1) \oplus E_8(-1)$ such that ι^* acts as above.

Given a free $\mathbf{Z}$-module M with an involution g, there is an isomorphism

$$(M,g) \cong M_1^s \oplus M_{-1}^t \oplus M_p^r,$$

for unique integers r, s, t (cf. [R]), where:

$$M_1 := (\mathbf{Z}, \iota_1 = 1), \qquad M_{-1} := (\mathbf{Z}, \iota_{-1} = -1), \qquad M_p := \left(\mathbf{Z}^2, \iota_p = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right).$$

Thus for a Nikulin involution acting on $H^2(X, \mathbf{Z})$ the invariants are $(s, t, r) = (6, 0, 8)$.

1.3. **The invariant lattice.** The invariant sublattice is:

$$H^2(X, \mathbf{Z})^\iota \cong \{(u, x, x) \in U^3 \oplus E_8(-1) \oplus E_8(-1)\} \cong U^3 \oplus E_8(-2).$$

The anti-invariant lattice is the lattice perpendicular to the invariant sublattice:

$$(H^2(X, \mathbf{Z})^\iota)^\perp \cong \{(0, x, -x) \in U^3 \oplus E_8(-1) \oplus E_8(-1)\} \cong E_8(-2).$$

The sublattices $H^2(X, \mathbf{Z})^\iota$ and $(H^2(X, \mathbf{Z})^\iota)^\perp$ are obviously primitive sublattices of $H^2(X, \mathbf{Z})$.

1.4. **The standard diagram.** The fixed point set of a Nikulin involution consists of exactly eight points ([Ni1, section 5]). Let $\beta : \tilde{X} \to X$ be the blow-up of X in the eight fixed points of ι. We denote by $\tilde{\iota}$ the involution on $\tilde{X}$ induced by ι. Moreover, let $\bar{Y} = X/\iota$ be the eight-nodal quotient of X, and let $Y = \tilde{X}/\tilde{\iota}$ be the minimal model of $\bar{Y}$, so Y is a K3 surface. This gives the 'standard diagram':

$$\begin{array}{ccc} X & \stackrel{\beta}{\longleftarrow} & \tilde{X} \\ \downarrow & & \downarrow \pi \\ \bar{Y} & \longleftarrow & Y. \end{array}$$

We denote by E_i, $i = 1, \ldots, 8$ the exceptional divisors in $\tilde{X}$ over the fixed points of ι in X, and by $N_i = \pi(E_i)$ their images in Y, these are (-2)-curves.

1.5. **The Nikulin lattice.** The minimal primitive sublattice of $H^2(Y, \mathbf{Z})$ containing the N_i is called the Nikulin lattice N (cf. [Mo, section 5]). As $N_i^2 = -2$, $N_i N_j = 0$ for $i \neq j$, the Nikulin lattice contains the lattice $< -2 >^8$. The lattice N has rank eight and is spanned by the N_i and a class $\hat{N}$:

$$N = \langle N_1, \ldots, N_8, \hat{N} \rangle, \qquad \hat{N} := (N_1 + \ldots + N_8)/2.$$

A set of 8 rational curves on a K3 surface whose sum is divisible by 2 in the Néron Severi group is called an even set, see [B] and section 3 for examples.

1.6. **The cohomology of $\tilde{X}$.** It is well-known that

$$H^2(\tilde{X}, \mathbf{Z}) \cong H^2(X, \mathbf{Z}) \oplus (\oplus_{i=1}^8 \mathbf{Z} E_i) \cong U^3 \oplus E_8(-1)^2 \oplus < -1 >^8 .$$

For a smooth surface S with torsion free $H^2(S, \mathbf{Z})$, the intersection pairing, given by the cup product to $H^4(S, \mathbf{Z}) = \mathbf{Z}$, gives an isomorphism $H^2(S, \mathbf{Z}) \to Hom_{\mathbf{Z}}(H^2(S, \mathbf{Z}), \mathbf{Z})$.

The map β^* is:

$$\beta^* : H^2(X, \mathbf{Z}) \longrightarrow H^2(\tilde{X}, \mathbf{Z}) = H^2(Y, \mathbf{Z}) \oplus (\oplus_{i=1}^8 \mathbf{Z} E_i), \qquad x \longmapsto (x, 0),$$

and its dual $\beta_* : H^2(\tilde{X}, \mathbf{Z}) \to H^2(X, \mathbf{Z})$ is $(x, e) \mapsto x$.

Let $\pi : \tilde{X} \to Y$ be the quotient map, let $\pi^* : H^2(Y, \mathbf{Z}) \to H^2(\tilde{X}, \mathbf{Z})$ be the induced map on the cohomology and let $\pi_* : H^2(\tilde{X}, \mathbf{Z}) \to H^2(Y, \mathbf{Z})$ be its dual, so:

$$\pi_* a \cdot b = a \cdot \pi^* b \qquad (a \in H^2(\tilde{X}, \mathbf{Z}),\ b \in H^2(Y, \mathbf{Z})).$$

Moreover, as π^* is compatible with cup product we have:

$$\pi^* b \cdot \pi^* c = 2(b \cdot c) \qquad (b, c \in H^2(Y, \mathbf{Z})).$$

1.7. **Lattices.** For a lattice $M := (M, b)$, where b is a $\mathbf{Z}$-valued bilinear form on a free $\mathbf{Z}$-module M, and an integer n we let $M(n) := (M, nb)$. In particular, M and $M(n)$ have the same underlying $\mathbf{Z}$-module, but the identity map $M \to M(n)$ is *not* an isometry unless $n = 1$ or $M = 0$.

1.8. **Proposition.** Using the notations and conventions as above, the map $\pi_* : H^2(\tilde{X}, \mathbf{Z}) \longrightarrow H^2(Y, \mathbf{Z})$ is given by

$$\pi_* : U^3 \oplus E_8(-1) \oplus E_8(-1) \oplus < -1 >^8 \longrightarrow U(2)^3 \oplus N \oplus E_8(-1) \ \hookrightarrow H^2(Y, \mathbf{Z}),$$

$$\pi_* :\ (u, x, y, z) \longmapsto (u, z, x + y).$$

The map π^*, on the sublattice $U(2)^3 \oplus N \oplus E_8(-1)$ of $H^2(Y, \mathbf{Z})$ is given by:

$$\pi^* : U(2)^3 \oplus N \oplus E_8(-1) \ \hookrightarrow\ H^2(\tilde{X}, \mathbf{Z}) \cong U^3 \oplus E_8(-1) \oplus E_8(-1) \oplus < -1 >^8,$$

$$\pi^* : (u, n, x) \longmapsto (2u, x, x, 2\tilde{n}),$$

here if $n = \sum n_i N_i$, $\tilde{n} = \sum n_i E_i$.

Proof. This follows easily from the results of Morrison. In the proof of [Mo, Theorem 5.7], it is shown that the image of each copy of $E_8(-1)$ under π_* is isomorphic to $E_8(-1)$. As $E_8(-1)$ is unimodular, it is a direct summand of the image of π_*. As $\pi_* \iota^* = \pi_*$, we get that $\pi_*(0, x, 0, 0) = \pi_*(0, 0, y, 0) \in E_8(-1)$. The $< -1 >^8$ maps into N (the image has index two). As U^3 is a direct summand of $H^2(X, \mathbf{Z})^\iota$, [Mo, Proposition 3.2] gives the first component.

As π_* and π^* are dual maps, $\pi^* a = b$ if for all $c \in H^2(\tilde{X}, \mathbf{Z})$ one has $(b \cdot c)_{\tilde{X}} = (a \cdot \pi_* c)_Y$. In particular, if $a \in U(2)^3$ and $c \in U^3$ we get $(\pi^* a \cdot c)_{\tilde{X}} = (a \cdot \pi_* c)_Y = 2(a \cdot c)_{\tilde{X}}$ since we compute in $U(2)^3$, hence $\pi^* a = 2a$. Similarly, $(\pi^* N_i \cdot E_j)_{\tilde{X}} = (N_i \cdot \pi_* E_j)_Y = -2\delta_{ij}$, so $\pi^* N_i = 2E_i$ (this also follows from the fact that the N_i are classes of the branch curves, so $\pi^* N_i$ is twice the class of $\pi^{-1}(N_i) = E_i$). Finally for $x \in E_8(-1)$ and $(y, 0) \in E_8(-1)^2$ we have $(\pi^* x \cdot (y, 0))_{\tilde{X}} = (x \cdot \pi_*(y, 0))_Y = (x \cdot y)_Y$ and also $(\pi^* x \cdot (0, y))_{\tilde{X}} = (x \cdot y)_Y$, so $\pi^* x = (x, x) \in E_8(-1)^2$. $\square$

1.9. **Extending π^*.** To determine the homomorphism $\pi^* : H^2(Y, \mathbf{Z}) \to H^2(\tilde{X}, \mathbf{Z})$ on all of $H^2(Y, \mathbf{Z})$, and not just on the sublattice of finite index $U(2)^3 \oplus N \oplus E_8(-1)$ we need to study the embedding $U(2)^3 \oplus N \hookrightarrow U^3 \oplus E_8(-1)$. This is done below. For any $x \in U^3 \oplus E_8(-1)$, one has $2x \in U(2)^3 \oplus N$ and $\pi^*(2x)$ determined as in Proposition 1.8. As π^* is a homomorphism and lattices are torsion free, one finds $\pi^* x$ as $\pi^* x = (\pi^*(2x))/2$.

1.10. **Lemma.** The sublattice of $(U(2)^3 \oplus N) \otimes \mathbf{Q}$ generated by $U(2)^3 \oplus N$ and the following six elements, each divided by two, is isomorphic to $U^3 \oplus E_8(-1)$:

$$\begin{array}{lll} e_1 + (N_1 + N_2 + N_3 + N_8), & e_2 + (N_1 + N_5 + N_6 + N_8), & e_3 + (N_2 + N_6 + N_7 + N_8), \\ f_1 + (N_1 + N_2 + N_4 + N_8), & f_2 + (N_1 + N_5 + N_7 + N_8), & f_3 + (N_3 + N_4 + N_5 + N_8), \end{array}$$

here e_i, f_i are the standard basis of the i-th copy of $U(2)$ in $U(2)^3$. Any embedding of $U(2)^3 \oplus N$ into $U^3 \oplus E_8(-1)$ such that the image of N is primitive in $U^3 \oplus E_8(-1)$ is isometric to this embedding.

Proof. The theory of embeddings of lattices can be found in [Ni2, section 1]. The dual lattice M^* of a lattice $M = (M, b)$ is

$$M^* = Hom(M, \mathbf{Z}) = \{x \in M \otimes \mathbf{Q} : \ b(x,m) \in \mathbf{Z} \ \ \forall m \in M\}.$$

Note that $M \hookrightarrow M^*$, intrinsically by $m \mapsto b(m, -)$ and concretely by $m \mapsto m \otimes 1$. If (M, b_M) and (L, b_L) are lattices such that $M \hookrightarrow L$, that is $b_M(m, m') = b_L(m, m')$ for $m, m' \in M$, then we have a map $L \to M^*$ by $l \mapsto b_L(l, -)$. In case M has finite index in L, so $M \otimes \mathbf{Q} \cong L \otimes \mathbf{Q}$, we get inclusions:

$$M \hookrightarrow L \hookrightarrow L^* \hookrightarrow M^*.$$

Therefore L is determined by the image of L/M in the finite group $A_M := M^*/M$, the discriminant group of M.

Since $b = b_M$ extends to a $\mathbf{Z}$-valued bilinear form on $L \subset M^*$ we get $q(l) := b_L(l, l) \in \mathbf{Z}$ for $l \in L$. If L is an even lattice, the discriminant form

$$q_M : A_M \longrightarrow \mathbf{Q}/2\mathbf{Z}, \qquad m^* \longmapsto b_L(m^*, m^*)$$

is identically zero on the subgroup $L/M \subset A_M$. In this way one gets a bijection between even overlattices of M and isotropic subgroups of A_M. In our case $M = K \oplus N$, with $K = U(2)^3$, so $A_M = A_K \oplus A_N$ and an isotropic subgroup of A_M is the direct sum of an isotropic subgroup of A_K and one isotropic subgroup of A_N. We will see that $(A_K, q_K) \cong (A_N, -q_N)$, hence the even unimodular overlattices L of M, with N primitive in L, correspond to isomorphisms $\gamma : A_N \to A_K$ with $q_N = -q_K \circ \gamma$. Then one has that

$$L/M = \{(\gamma(\bar{n}), \bar{n}) \in A_M = A_K \oplus A_N : \ \bar{n} \in A_N\}.$$

The overlattice L_γ corresponding to γ is:

$$L_\gamma := \{(u, n) \in K^* \oplus N^* : \ \gamma(\bar{n}) = \bar{u} \ \}.$$

We will show that the isomorphism γ is unique up to isometries of K and N.
Let e, f be the standard basis of U, so $e^2 = f^2 = 0, ef = 1$, then $U(2)$ has the same basis with $e^2 = f^2 = 0, ef = 2$. Thus $U(2)^*$ has basis $e/2, f/2$ with $(e/2)^2 = (f/2)^2 = 0, (e/2)(f/2) = 2/4 = 1/2$. Thus $A_K = (U(2)^*/U(2))^3 \cong (\mathbf{Z}/2\mathbf{Z})^6$, and the discriminant form q_K on A_K is given by

$$q_K : A_K = (\mathbf{Z}/2\mathbf{Z})^6 \longrightarrow \mathbf{Z}/2\mathbf{Z}, \qquad q_K(x) = x_1x_2 + x_3x_4 + x_5x_6.$$

The Nikulin lattice N contains $\oplus \mathbf{Z}N_i$ with $N_i^2 = -2$, hence $N^* \subset \mathbf{Z}(N_i/2)$. As $N =< N_i, (\sum N_i)/2 >$ we find that $n^* \in \mathbf{Z}(N_i/2)$ is in N^* iff $n^* \cdot (\sum N_i)/2 \in \mathbf{Z}$, that is, $n^* = \sum x_i(N_i/2)$ with $\sum x_i \equiv 0 \bmod 2$. Thus we obtain an identification:

$$A_N = N^*/N = \{(x_1, \ldots, x_8) \in (\mathbf{Z}/2\mathbf{Z})^8 : \sum x_i = 0\}/ < (1, \ldots, 1) > \cong (\mathbf{Z}/2\mathbf{Z})^6,$$

where $(1, \ldots, 1)$ is the image of $(\sum N_i)/2$. Any element in A_N has a unique representative which is either 0, $(N_i + N_j)/2$, with $i \neq j$ and $((N_i + N_j)/2)^2 = 1 \bmod 2\mathbf{Z}$, or $(N_1 + N_i + N_j + N_k)/2$ $(= (N_l + N_m + N_n + N_r)/2)$, with distinct indices and with $\{i, \ldots, r\} = \{2, \ldots, 8\}$ and $((N_1 + N_i + N_j + N_k)/2)^2 = 0 \bmod 2$. The quadratic spaces, over the field $\mathbf{Z}/2\mathbf{Z}$, $((\mathbf{Z}/2\mathbf{Z})^6, q_K)$ and $((\mathbf{Z}/2\mathbf{Z})^6, q_N)$ are isomorphic, an explicit isomorphism is defined by

$$\gamma : A_N \longrightarrow A_K, \qquad \gamma((N_1 + N_2 + N_3 + N_8)/2) = e_1/2,$$

etc. where we use the six elements listed in the lemma.

The orthogonal group of the quadratic space $((\mathbf{Z}/2\mathbf{Z})^6, q_N)$ obviously contains S_8, induced by permutations of the basis vectors in $(\mathbf{Z}/2\mathbf{Z})^8$, and these groups are actually equal cf. [Co]. Thus any two isomorphisms $A_N \to A_K$ preserving the quadratic forms differ by an isometry of A_N which is induced by a permutation of the nodal classes $N_1, \ldots, N_8$. A permutation of the 8 nodal curves N_i in N obviously extends to an isometry of N.

This shows that such an even unimodular overlattice of $U(2)^3 \oplus N$ is essentially unique. As these are classified by their rank and signature, the only possible one is $U^3 \oplus E_8(-1)$. Using the isomorphism γ, one obtains the lattice L_γ, which is described in the lemma. $\square$

1.11. **The lattices $N \oplus N$ and Γ_{16}.** Using the methods of the proof of Lemma 1.10 we show that any even unimodular overlattice L of $N \oplus N$ such that $N \oplus \{0\}$ is primitive in L, is isomorphic to $\Gamma_{16}(-1)$ (cf. [Se, Chapter V, 1.4.3]). The lattice $\Gamma_{16}(-1)$ is the unique even unimodular negative definite lattice which is not generated by its roots, i.e. by vectors v with $v^2 = -2$.

The discriminant form q_N of the lattice N has values in $\mathbf{Z}/2\mathbf{Z}$, hence $q_N = -q_N$. Therefore isomorphisms $\gamma : N \to N$ correspond to the even unimodular overlattices L_γ of $N \oplus N$ with $N \oplus \{0\}$ primitive in L_γ. Since $N \oplus N$ is negative definite, so is L_γ. The uniqueness of the overlattice follows, as before, from the fact $O(q_N) \cong S_8$. To see that this overlattice is $\Gamma_{16}(-1)$, recall that

$$\Gamma_{16} = \{x = (x_1, \ldots, x_{16}) \in \mathbf{Q}^{16} : \ 2x_i \in \mathbf{Z}, \ x_i - x_j \in \mathbf{Z}, \ \sum x_i \in 2\mathbf{Z} \},$$

and the bilinear form on Γ_{16} is given by $\sum x_i y_i$. Let e_i be the standard basis vectors of $\mathbf{Q}^{16}$. As

$$N \oplus N \hookrightarrow \Gamma_{16}(-1), \qquad (N_i, 0) \longmapsto e_i + e_{i+8}, \qquad (0, N_i) \longmapsto e_i - e_{i+8},$$

is a primitive embedding $N \oplus N$ into $\Gamma_{16}(-1)$ (note $(\hat{N}, 0) \mapsto (\sum e_i)/2 \in \Gamma_{16}$, $(0, \hat{N}) \mapsto ((\sum_{i=1}^{8} e_i) - (\sum_{i=9}^{16} e_i))/2 \in \Gamma_{16})$ the claim follows.

2. Eleven dimensional families of K3 surfaces with a Nikulin involution

2.1. **Néron Severi groups.** As X is a K3 surface it has $H^{1,0}(X) = 0$ and

$$Pic(X) = NS(X) = H^{1,1}(X) \cap H^2(X, \mathbf{Z}) = \{x \in H^2(X, \mathbf{Z}) : \ x \cdot \omega = 0 \ \forall \omega \in H^{2,0}(X)\}.$$

For $x \in (H^2(X, \mathbf{Z})^\iota)^\perp$ we have $\iota^* x = -x$. As $\iota^*\omega = \omega$ for $\omega \in H^{2,0}(X)$ we get:

$$\omega \cdot x = \iota^*\omega \cdot \iota^* x = -\omega \cdot x \qquad \text{hence} \quad (H^2(X, \mathbf{Z})^\iota)^\perp \subset NS(X).$$

As we assume X to be algebraic, there is a very ample line bundle M on X, so $M \in NS(X)$ and $M^2 > 0$. Therefore the Néron Severi group of X contains $E_8(-2) \cong (H^2(X, \mathbf{Z})^\iota)^\perp$ as a primitive sublattice and has rank at least 9.

The following proposition gives all even, rank 9, lattices of signature $(1+, 8-)$ which contain $E_8(-2)$ as a primitive sublattice. We will show in Proposition 2.3 that any of these lattices is the Néron Severi group of a K3 surface with a Nikulin involution. Moreover, the moduli space of K3 surfaces, which contain such a lattice in the Néron Severi group, is an 11-dimensional complex variety.

2.2. **Proposition.** Let X be a K3 surface with a Nikulin involution ι and assume that the Néron Severi group of X has rank 9. Let L be a generator of $E_8(-2)^\perp \subset NS(X)$ with $L^2 = 2d > 0$ and let

$$\Lambda = \Lambda_{2d} := \mathbf{Z}L \oplus E_8(-2) \quad (\subset NS(X)).$$

Then we may assume that L is ample and:

1. in case $L^2 \equiv 2 \bmod 4$ we have $\Lambda = NS(X)$;
2. in case $L^2 \equiv 0 \bmod 4$ we have that either $NS(X) = \Lambda$ or $NS(X) \cong \tilde{\Lambda}$ where $\tilde{\Lambda} = \Lambda_{\widetilde{2d}}$ is the unique even lattice containing Λ with $\tilde{\Lambda}/\Lambda \cong \mathbf{Z}/2\mathbf{Z}$ and such that $E_8(-2)$ is a primitive sublattice of $\tilde{\Lambda}$.

Proof. As $L^2 > 0$, either L or $-L$ is effective, so may assume that L is effective. As there are no (-2)-curves in $L^\perp = E_8(-2)$, any (-2)-curve N has class $aL + e$ with $a \in \mathbf{Z}_{>0}$ and $e \in E_8(-2)$. Thus $NL = aL^2 > 0$ and therefore L is ample.

From the definition of L and the description of the action of ι on $H^2(X, \mathbf{Z})$ it follows that $\mathbf{Z}L$ and $E_8(-2)$ respectively are primitive sublattices of $NS(X)$. The discriminant group of $< L >$ is $A_L := < L >^* / < L > \cong \mathbf{Z}/2d\mathbf{Z}$ with generator $(1/2d)L$ where $L^2 = 2d$ and thus $q_L((1/2d)L) = 1/2d$. The discriminant group of $E_8(-2)$ is $A_E \cong (1/2)E_8(-2)/E_8(-2) \cong (\mathbf{Z}/2\mathbf{Z})^8$, as the quadratic form on $E_8(-2)$ takes values in $4\mathbf{Z}$, the discriminant form q_E takes values in $\mathbf{Z}/2\mathbf{Z}$.

The even lattices $\tilde{\Lambda}$ which have Λ as sublattice of finite index correspond to isotropic subgroups H of $A_L \oplus A_E$ where $A_L := < L >^* / < L > \cong \mathbf{Z}/2d\mathbf{Z}$. If $E_8(-2)$ is a primitive sublattice of $\tilde{\Lambda}$, H must have trivial intersection with both A_L and A_E. Since A_E is two-torsion, it follows that H is generated by $((1/2)L, v/2)$ for some $v \in E_8(-2)$. As $((1/2)L)^2 = d/2 \bmod 2\mathbf{Z}$ and $(v/2)^2 \in \mathbf{Z}/2\mathbf{Z}$, for H to be isotropic, d must be even. Moreover, if $d = 4m + 2$ we must have $v^2 = 8k + 4$ for some k and if $d = 4m$ we must have $v^2 = 8k$. Conversely, such a $v \in E_8(-2)$ defines an isotropic subgroup $< (L/2, v/2) > \subset A_L \oplus A_E$ which corresponds to an overlattice $\tilde{\Lambda}$.

The group $O(E_8(-2))$ contains $W(E_8)$ (cf. [Co]) which maps onto $O(q_E)$. As $O(q_E)$ has three orbits on A_E, they are $\{0\}$, $\{v/2 : (v/2)^2 \equiv 0\ (2)\}$ and $\{v/2 : (v/2)^2 \equiv 1\ (2)\}$, the overlattice is unique up to isometry. $\square$

2.3. **Proposition.** Let $\Gamma = \Lambda_{2d}$, $d \in \mathbf{Z}_{>0}$ or $\Gamma = \Lambda_{\widetilde{2d}}$, $d \in 2\mathbf{Z}_{>0}$. Then there exists a K3 surface X with a Nikulin involution ι such that $NS(X) \cong \Gamma$ and $(H^2(X,\mathbf{Z})^\iota)^\perp \cong E_8(-2)$.

The coarse moduli space of Γ-polarized K3 surfaces has dimension 11 and will be denoted by $\mathcal{M}_{2d}$ if $\Gamma = \Lambda_{2d}$ and by $\mathcal{M}_{\widetilde{2d}}$ if $\Gamma = \Lambda_{\widetilde{2d}}$.

Proof. We show that there exists a K3 surface X with a Nikulin involution ι such that $NS(X) \cong \Lambda_{\widetilde{2d}}$ and under this isomorphism $(H^2(X,\mathbf{Z})^\iota)^\perp \cong E_8(-2)$. The case $NS(X) \cong \Lambda_{2d}$ is similar but easier and is left to the reader.

The primitive embedding of $\Lambda_{\widetilde{2d}}$ in the unimodular lattice $U^3 \oplus E_8(-1)^2$ is unique up to isometry by [Ni2, Theorem 1.14.1], and we will identify $\Lambda_{\widetilde{2d}}$ with a primitive sublattice of $U^3 \oplus E_8(-1)^2$ from now on. We choose an $\omega \in \Lambda_{\widetilde{2d}}^\perp \otimes_{\mathbf{Z}} \mathbf{C}$ with $\omega^2 = 0$, $\omega\bar{\omega} > 0$ and general with these properties, hence $\omega^\perp \cap (U^3 \oplus E_8(-1)^2) = \Lambda_{\widetilde{2d}}$. By the 'surjectivity of the period map', there exists a K3 surface X with an isomorphism $H^2(X,\mathbf{Z}) \cong U^3 \oplus E_8(-1)^2$ such that $NS(X) \cong \Lambda_{\widetilde{2d}}$.

The involution of $\Lambda = \mathbf{Z}L \oplus E_8(-2)$ which is trivial on L and -1 on $E_8(-2)$, extends to an involution of $\Lambda_{\widetilde{2d}} = \Lambda + \mathbf{Z}(L/2, v/2)$. The involution is trivial on the discriminant group of $\Lambda_{\widetilde{2d}}$ which is isomorphic to $(\mathbf{Z}/2\mathbf{Z})^6$. Therefore it extends to an involution ι_0 of $U^3 \oplus E_8(-1)^2$ which is trivial on $\Lambda_{\widetilde{2d}}^\perp$. As $((U^3 \oplus E_8(-1)^2)^{\iota_0})^\perp = E_8(-2)$ is negative definite, contains no (-2)-classes and is contained in $NS(X)$, results of Nikulin ([Ni1, Theorems 4.3, 4.7, 4.15]) show that X has a Nikulin involution ι such that $\iota^* = \iota_0$ up to conjugation by an element of the Weyl group of X. Since we assume L to be ample and the ample cone is a fundamental domain for the Weyl group action, we do get $\iota^* = \iota_0$, hence $(H^2(X,\mathbf{Z})^\iota)^\perp \cong E_8(-2)$.

For the precise definition of Γ-polarized K3 surfaces we refer to [Do]. We just observe that each point of the moduli space corresponds to a K3 surface X with a primitive embedding $\Gamma \hookrightarrow NS(X)$. The moduli space is a quotient of the 11-dimensional domain

$$\mathcal{D}_\Gamma = \{\omega \in \mathbf{P}(\Gamma^\perp \otimes_{\mathbf{Z}} \mathbf{C}) :\ \omega^2 = 0,\ \omega\bar{\omega} > 0\,\}$$

by an arithmetic subgroup of $O(\Gamma)$. $\square$

2.4. **Note on the Hodge conjecture.** For a smooth projective surface S with torsion free $H^2(S,\mathbf{Z})$, let $T_S := NS(S)^\perp \subset H^2(S,\mathbf{Z})$ and let $T_{S,\mathbf{Q}} = T_S \otimes_{\mathbf{Z}} \mathbf{Q}$. Then T_S, the transcendental lattice of S, is an (integral, polarized) weight two Hodge structure.

The results in section 1 show that $\pi_* \circ \beta^*$ induces an isomorphism of rational Hodge structures:

$$\phi_\iota : T_{X,\mathbf{Q}} \xrightarrow{\cong} T_{Y,\mathbf{Q}},$$

in fact, both are isomorphic to $T_{\tilde{X},\mathbf{Q}}$. Any homomorphism of rational Hodge structures $\phi : T_{X,\mathbf{Q}} \to T_{Y,\mathbf{Q}}$ defines, using projection and inclusion, a map of Hodge structures $H^2(X,\mathbf{Q}) \to T_{X,\mathbf{Q}} \to T_{Y,\mathbf{Q}} \hookrightarrow H^2(Y,\mathbf{Q})$ and thus it gives a Hodge (2,2)-class

$$\phi \in H^2(X,\mathbf{Q})^* \otimes H^2(Y,\mathbf{Q}) \cong H^2(X,\mathbf{Q}) \otimes H^2(Y,\mathbf{Q}) \hookrightarrow H^4(X \times Y,\mathbf{Q}),$$

where we use Poincaré duality and the Künneth formula. Obviously, the isomorphism $\phi_\iota : T_{X,\mathbf{Q}} \to T_{Y,\mathbf{Q}}$ corresponds to the class of the codimension two cycle which is the image of $\tilde{X}$ in $X \times Y$ under (β, π).

Mukai showed that any homomorphism between $T_{S,\mathbf{Q}}$ and $T_{Z,\mathbf{Q}}$ where S and Z are K3 surfaces which is moreover an isometry (w.r.t. the quadratic forms induced by the intersection forms) is induced by an algebraic cycle if $\dim T_{S,\mathbf{Q}} \le 11$ ([Mu, Corollary 1.10]). Nikulin, [Ni3, Theorem 3], strengthened this result and showed that it suffices that $NS(X)$ contains a class e with $e^2 = 0$. In particular, this implies that any Hodge isometry $T_{S,\mathbf{Q}} \to T_{Z,\mathbf{Q}}$ is induced by an algebraic cycle if $\dim T_{S,\mathbf{Q}} \le 18$ (cf. [Ni3, proof of Theorem 3]).

The Hodge conjecture predicts that any homomorphism of Hodge structures between $T_{S,\mathbf{Q}}$ and $T_{Z,\mathbf{Q}}$ is induced by an algebraic cycle, *without* requiring that it is an isometry. There are few results in this direction, it is therefore maybe worth noticing that ϕ_ι is not an isometry if T_X has odd rank, see the proposition below. In [GL] a similar result of D. Morrison in a more special case is used to obtain new results on the Hodge conjecture. In Proposition 4.2 we show that there exists a K3 surface X with Nikulin involution where $T_{X,\mathbf{Q}}$ has even rank and $T_{X,\mathbf{Q}}$ *is* isometric to $T_{Y,\mathbf{Q}}$.

2.5. **Proposition.** Let $\phi_\iota : T_{X,\mathbf{Q}} \xrightarrow{\cong} T_{Y,\mathbf{Q}}$ be the isomorphism of Hodge structures induced by the Nikulin involution ι on X and assume that $\dim T_{X,\mathbf{Q}}$ is an odd integer. Then ϕ_ι is not an isometry.

Proof. Let $Q : \mathbf{Q}^n \to \mathbf{Q}$ be a quadratic form, then Q is defined by an $n \times n$ symmetric matrix, which we also denote by Q: $Q(x) := {}^t x Q x$. An isomorphism $A : \mathbf{Q}^n \to \mathbf{Q}^n$ gives an isometry between $(\mathbf{Q}^n, Q)$ and $(\mathbf{Q}^n, Q')$ iff $Q' = {}^t A^{-1} Q A^{-1}$). In particular, if $(\mathbf{Q}^n, Q) \cong (\mathbf{Q}^n, Q')$ the quotient $\det(Q)/\det(Q')$ must be a square in $\mathbf{Q}^*$.

For a $\mathbf{Z}$-module M we let $M_\mathbf{Q} := M \otimes_\mathbf{Z} \mathbf{Q}$. Let V_X be the orthogonal complement of $E_8(-2)_\mathbf{Q} \subset NS(X)_\mathbf{Q}$, then $\det(NS(X)_\mathbf{Q}) = 2^8 \det(V_X)$ up to squares. Let V_Y be the orthogonal complement of $N_\mathbf{Q} \subset NS(Y)_\mathbf{Q}$ then $\det(NS(Y)_\mathbf{Q}) = 2^6 \det(V_Y)$ up to squares. Now $\beta_*\pi^* : H^2(Y, \mathbf{Q}) \to H^2(X, \mathbf{Q})$ induces an isomorphism $V_X \to V_Y$ which satisfies $(\beta_*\pi^*x)(\beta_*\pi^*y) = 2xy$ for $x, y \in V_Y$, hence $\det(V_X) = 2^d \det(V_Y)$ where $d = \dim V_X = 22 - 8 - \dim T_{X,\mathbf{Q}}$, so d is odd by assumption.

For a K3 surface S, $\det(T_{S,\mathbf{Q}}) = -\det(NS(S)_\mathbf{Q})$ and thus $\det(T_{X,\mathbf{Q}})/\det(T_{Y,\mathbf{Q}}) = 2^{d+2}$ up to squares. As d is odd and 2 is not a square in the multiplicative group of $\mathbf{Q}$, it follows that there exists no isometry between $T_{X,\mathbf{Q}}$ and $T_{Y,\mathbf{Q}}$. $\square$

2.6. **The bundle L.** In case $NS(X)$ has rank 9, the ample generator L of $E_8(-2)^\perp \subset NS(X)$ defines a natural map

$$\phi_L : X \longrightarrow \mathbf{P}^g, \qquad g = h^0(L) - 1 = L^2/2 + 1$$

which we will use to study X and Y. As $\iota^* L \cong L$, the involution ι acts as an involution on $\mathbf{P}^g = |L|^*$ and thus it has two fixed spaces $\mathbf{P}^a, \mathbf{P}^b$ with $(a+1) + (b+1) = g + 1$. The fixed points of ι map to these fixed spaces. Even though L is ι-invariant, it is not the case in general that on $\tilde{X}$ we have $\beta^* L = \pi^* M$ for some line bundle $M \in NS(Y)$. In fact, $\beta^* L = \pi^* M$ implies

$L^2 = (\beta^* L)^2 = (\pi^* M)^2 = 2M^2$ and as M^2 is even we get $L^2 \in 4\mathbf{Z}$. Thus if $L^2 \notin 4\mathbf{Z}$, the ι-invariant line bundle L cannot be obtained by pull-back from Y. On the other hand, if for example $|L|$ contains a reduced ι-invariant divisor D which does not pass through the fixed points, then $\beta^* D = \beta^{-1} D$ is invariant under $\tilde{\iota}$ on $\tilde{X}$ and does not contain any of the E_i as a component. Then $\beta^* D = \pi^* D'$ where $D' \subset Y$ is the reduced divisor with support $\pi(\beta^{-1} D)$.

The following lemma collects the basic facts on L and the splitting of $\mathbf{P}^g = \mathbf{P}H^0(X, L)^*$.

2.7. **Proposition.**

(1) Assume that $NS(X) = \mathbf{Z}L \oplus E_8(-2)$. Let $E_1, \dots, E_8$ be the exceptional divisors on $\tilde{X}$.
In case $L^2 = 4n+2$, there exist line bundles $M_1, M_2 \in NS(Y)$ such that for a suitable numbering of these E_i we have:

$$\beta^* L - E_1 - E_2 = \pi^* M_1, \qquad \beta^* L - E_3 - \ldots - E_8 = \pi^* M_2.$$

The decomposition of $H^0(X, L)$ into ι^*-eigenspaces is:

$$H^0(X, L) \cong \pi^* H^0(Y, M_1) \oplus \pi^* H^0(Y, M_2), \qquad (h^0(M_1) = n + 2,\ h^0(M_2) = n + 1).$$

and the eigenspaces $\mathbf{P}^{n+1}, \mathbf{P}^n$ contain six, respectively two, fixed points.
In case $L^2 = 4n$, for a suitable numbering of the E_i we have:

$$\beta^* L - E_1 - E_2 - E_3 - E_4 = \pi^* M_1, \qquad \beta^* L - E_5 - E_6 - E_7 - E_8 = \pi^* M_2$$

with $M_1, M_2 \in NS(Y)$. The decomposition of $H^0(X, L)$ into ι^*-eigenspaces is:

$$H^0(X, L) \cong \pi^* H^0(Y, M_1) \oplus \pi^* H^0(Y, M_2), \qquad (h^0(M_1) = h^0(M_2) = n + 1).$$

and each of the eigenspaces $\mathbf{P}^n$ contains four fixed points.

(2) Assume that $\mathbf{Z}L \oplus E_8(-2)$ has index two in $NS(X)$. Then there is a line bundle $M \in NS(Y)$ such that:

$$\beta^* L \cong \pi^* M, \qquad H^0(X, L) \cong H^0(Y, M) \oplus H^0(Y, M - \hat{N}),$$

where $\hat{N} = (\sum_{i=1}^8 N_i)/2 \in NS(Y)$ and this is the decomposition of $H^0(X, L)$ into ι^*-eigenspaces. One has $h^0(M) = n + 2$, $h^0(M - \hat{N}) = n$, and all fixed points map to the eigenspace $\mathbf{P}^{n+1} \subset \mathbf{P}^{2n+1} = \mathbf{P}^g$.

Proof. The primitive embedding of $\mathbf{Z}L \oplus E_8(-2)$ in the unimodular lattice $U^3 \oplus E_8(-1)^2$ is unique up to isometry by [Ni2, Theorem 1.14.1]. Therefore if $L^2 = 2r$ we may assume that $L = e_1 + rf_1 \in U \subset U^3 \oplus E_8(-1)^3$ where e_1, f_1 are the standard basis of the first copy of U.

In case $r = 2n+1$, it follows from Lemma 1.10 that $(e_1 + (2n+1)f_1 + N_3 + N_4)/2 \in NS(Y)$. By Proposition 1.8, $M_1 := (e_1+(2n+1)f_1+N_3+N_4)/2-N_3-N_4$ satisfies $\pi^* M_1 = \beta^* L-E_3-E_4$. Similarly, let $M_2 = (e_1 + (2n + 1)f_1 + N_3 + N_4)/2 - \hat{N} \in NS(Y)$, then $\pi^* M_2 = \beta^* L - (E_1 + E_2 + E_5 + \ldots + E_8)$.

Any two sections $s, t \in H^0(X, L)$ lie in the same ι^*-eigenspace iff the rational function $f = s/t$ is ι-invariant. Thus $s, t \in \pi^* H^0(Y, M_i)$ are ι^*-invariant, hence each of these two spaces is contained in an eigenspace of ι^* in $H^0(X, L)$. If both are in the same eigenspace, then this eigenspace would have a section with no zeroes in the 8 fixed points of ι. But a ι-invariant divisor on X which doesn't pass through any fixed point is the pull back of divisor

on Y, which contradicts that L^2 is not a multiple of 4. Thus the $\pi^*H^0(Y,M_i)$ are in distinct eigenspaces. A dimension count shows that $h^0(L)=h^0(M_1)+h^0(M_2)$, hence the $\pi^*H^0(Y,M_i)$ are the eigenspaces.

In case $r=2n$, again by Lemma 1.10 we have $(e_1+N_1+N_2+N_3+N_8)/2\in NS(Y)$. Let $M_1:=nf_1+(e_1+N_1+N_2+N_3+N_8)/2-(N_1+N_2+N_3+N_8)$ then $\pi^*M_1=\beta^*L-(E_1+E_2+E_3+E_8)$. Put $M_2=M_1+\hat{N}-(N_4+N_5+N_6+N_7)$, then $\pi^*M_2=\beta^*L-(E_4+E_5+E_6+E_7)$. As above, the $\pi^*H^0(Y,M_i)$, $i=1,2$, are contained in distinct eigenspaces and a dimension count again shows that $h^0(L)=h^0(M_1)+h^0(M_2)$.

If $\mathbf{Z}L\oplus E_8(-2)$ has index two in $NS(X)$, the (primitive) embedding of $NS(X)$ into $U^3\oplus E_8(-1)$ is still unique up to isometry. Let $L^2=4n$. Choose an $\alpha\in E_8(-1)$ with $\alpha^2=-2$ if n is odd, and $\alpha^2=-4$ if n is even. Let $v=(0,\alpha,-\alpha)\in E_8(-2)\subset U^3\oplus E_8(-1)^2$ and let $L=(2u,\alpha,\alpha)\in U^3\oplus E_8(-1)^2$ where $u=e_1+(n+1)/2f_1$ if n is odd and $u=e_1+(n/2+1)f_1$ if n is even. Note that $L^2=4u^2+2\alpha^2=4n$ and that $(L+v)/2=(u,\alpha,0)\in U^3\oplus E_8(-1)^2$. Thus we get a primitive embedding of $NS(X)\hookrightarrow U^3\oplus E_8(-1)^2$ which extends the standard one of $E_8(-2)\subset NS(X)$. Proposition 1.8 shows that $\beta^*L=\pi^*M$ with $M=(u,0,\alpha)\in U^3(2)\oplus N\oplus E_8(-1)\subset H^2(Y,\mathbf{Z})$. For the double cover $\pi:\tilde{X}\to Y$ branched along $2\hat{N}=\sum N_i$ we have as usual: $\pi_*\mathcal{O}_{\tilde{X}}=\mathcal{O}_Y\oplus\mathcal{O}_Y(-\hat{N})$ hence, using the projection formula:

$$H^0(\tilde{X},\pi^*M)\cong H^0(Y,\pi_*(\pi^*M\otimes\mathcal{O}_{\tilde{X}})\cong H^0(Y,M)\oplus H^0(Y,M-\hat{N}).$$

Note that the sections in $\pi^*H^0(Y,M-\hat{N})$ vanish on all the exceptional divisors, hence the fixed points of ι map to a $\mathbf{P}^{n+1}$. $\square$

3. Examples

3.1. In Proposition 2.3 we showed that K3 surfaces with a Nikulin involution are parametrized by eleven dimensional moduli spaces $\mathcal{M}_{2d}$ and $\mathcal{M}_{\widetilde{4e}}$ with $d,e\in\mathbf{Z}_{>0}$. For some values of d,e we will now work out the geometry of the corresponding K3 surfaces. We will also indicate how to verify that the moduli spaces are indeed eleven dimensional.

3.2. **The case $\mathcal{M}_2$.** Let X be a K3 surface with Nikulin involution ι and $NS(X)\cong\mathbf{Z}L\oplus E_8(-2)$ with $L^2=2$ and $\iota^*L\cong L$ (cf. Proposition 2.3). The map $\phi_L:X\to\mathbf{P}^2$ is a double cover of $\mathbf{P}^2$ branched over a sextic curve C, which is smooth since there are no (-2)-curves in $L^\perp$. The covering involution will be denoted by $i:X\to X$. The fixed point locus of i is isomorphic to C.

As i^* is $+1$ on $\mathbf{Z}L$, -1 on $E_8(-2)$ and -1 on T_X, whereas ι^* is $+1$ on $\mathbf{Z}L$, -1 on $E_8(-2)$ and $+1$ on T_X, these two involutions commute. Thus ι induces an involution $\bar{\iota}_{\mathbf{P}^2}$ on $\mathbf{P}^2$ (which is ι^* acting on $\mathbf{P}H^0(X,L)^*$) and in suitable coordinates:

$$\bar{\iota}_{\mathbf{P}^2}:\ (x_0:x_1:x_2)\longmapsto(-x_0:x_1:x_2).$$

We have a commutative diagram

$$\begin{array}{ccccc} C & \hookrightarrow & X & \xrightarrow{\iota} & X \\ \cong\downarrow & & \downarrow\phi & & \downarrow\phi \\ C & \hookrightarrow & \mathbf{P}^2 & \xrightarrow{\bar{\iota}_{\mathbf{P}^2}} & \mathbf{P}^2=X/i. \end{array}$$

The fixed points of $\bar{\iota}_{\mathbf{P}^2}$ are:

$$(\mathbf{P}^2)^{\bar{\iota}_{\mathbf{P}^2}} = l_0 \cup \{p\}, \qquad l_0 :\ x_0 = 0, \quad p = (1:0:0).$$

The line l_0 intersects the curve C in six points, which are the images of six fixed points $x_3, \ldots, x_8$ of ι on X. Thus the involution ι induces an involution on $C \subset X$ with six fixed points. The other two fixed points x_1, x_2 of ι map to the point p, so i permutes these two fixed points of ι. In particular, these two points are not contained in C so $p \notin C$ $(\subset \mathbf{P}^2)$, which will be important in the moduli count below. The inverse image $C_2 = \phi^{-1}(l_0)$ is a genus two curve in the system $|L|$. Both ι and i induce the hyperelliptic involution on C_2. By doing then the quotient by ι, since this has six fixed points on C_2 we obtain a rational curve C_0.

To describe the eight nodal surface $\bar{Y} = X/\iota$, we use the involution $\bar{i}_{\bar{Y}}$ of $\bar{Y}$ which is induced by $i \in Aut(X)$. Then we have:

$$Q := \bar{Y}/\bar{i}_{\bar{Y}} \ \cong\ X/<\iota, i> \cong\ \mathbf{P}^2/\bar{\iota}_{\mathbf{P}^2}.$$

This leads to the following diagrams of double covers and fixed point sets:

$$\begin{tikzcd}[column sep=small] & X \arrow[dl] \arrow[d] \arrow[dr] & \\ \mathbf{P}^2 \arrow[dr] & \bar{Y} \arrow[d] & S \arrow[dl] \\ & Q & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & \{x_1, x_2\} \cup C \cup C_2 \arrow[dl] \arrow[d] \arrow[dr] & \\ \{p\} \cup C \cup l_0 \arrow[dr] & \{y_1, y_2\} \cup C_4 \cup C_0 \arrow[d] & \{p_9\} \cup C_4 \cup D_0 \arrow[dl] \\ & \{q\} \cup C_4 \cup H_0 & \end{tikzcd}$$

The quotient of $\mathbf{P}^2 = X/i$ by $\bar{\iota}_{\mathbf{P}^2}$ is isomorphic to a quadric cone Q in $\mathbf{P}^3$ whose vertex q is the image of the fixed point $(1:0:0)$. In coordinates, the quotient map is:

$$\mathbf{P}^2 \longrightarrow Q = \mathbf{P}^2/\bar{\iota}_{\mathbf{P}^2} \ \subset \mathbf{P}^3, \qquad (x_0 : x_1 : x_2) \longmapsto (y_0 : \ldots : y_3) = (x_0^2 : x_1^2 : x_1x_2 : x_2^2)$$

and Q is defined by $y_1y_3 - y_2^2 = 0$.

The sextic curve $C \subset \mathbf{P}^2$, which has genus 10, is mapped 2:1 to a curve C_4 on the cone. The double cover $C \to C_4$ ramifies in the six points where C intersects the line $x_0 = 0$. Thus the curve C_4 is smooth, has genus four and degree six (the plane sections of C_4 are the images of certain conic sections of the branch sextic) and does not lie in a plane (so C_4 spans $\mathbf{P}^3$). The only divisor class D of degree $2g-2$ with $h^0(D) \geq g$ on a smooth curve of genus g is the canonical class, hence C_4 is a canonically embedded curve. The image of the line l_0 is the plane section $H_0 \subset Q$ defined by $y_0 = 0$.

The branch locus in Q of the double cover

$$\bar{Y} \longrightarrow Q = \bar{Y}/\bar{i}_{\bar{Y}}$$

is the union of two curves, C_4 and the plane section H_0, these curves intersect in six points, and the vertex q of Q.

To complete the diagram, we consider the involution

$$j := \iota \circ i : X \longrightarrow X, \qquad S := X/j.$$

The fixed point set of j is the (smooth) genus two curve C_2 lying over the line l_0 in $\mathbf{P}^2$ (use $j(p) = p$ iff $\iota(p) = i(p)$ and consider the image of p in $\mathbf{P}^2$). Thus the quotient surface S is a smooth surface. The Riemann-Hurwitz formula implies that the image of C_2 in S is a curve $D_0 \in |-2K_S|$, note that $D_0 \cong C_2$.

The double cover $S \to Q$ branches over the curve $C_4 \subset Q$ and the vertex $q \in Q$. It is well-known that such a double cover is a Del Pezzo surface of degree 1 ([Dem], [DoO]) and the map $S \to Q \subset \mathbf{P}^3$ is given by ϕ_{-2K}, which verifies that the image of D_0 is a plane section.

On the other hand, any Del Pezzo surface of degree 1 is isomorphic to the blow up of $\mathbf{P}^2$ in eight points. The linear system $|-K_S|$ corresponds to the pencil of elliptic curves on the eight points, the ninth base point in $\mathbf{P}^2$ corresponds to the unique base point p_9 of $|-K_S|$ in S. The point p_9 maps to the vertex $q \in Q$ under the 2:1 map ϕ_{-2K} ([DoO, p. 125]). The Néron Severi group of S is thus isomorphic to

$$NS(S) \cong \mathbf{Z}e_0 \oplus \mathbf{Z}e_1 \oplus \ldots \oplus \mathbf{Z}e_8, \qquad e_0^2 = 1, \quad e_i^2 = -1 \quad (1 \le i \le 8)$$

and $e_ie_j = 0$ if $i \neq j$. The canonical class is $K_S = -3e_0 + e_1 + \ldots + e_8$. Since $K_S^2 = 1$, we get a direct sum decomposition:

$$NS(S) \cong \mathbf{Z}K_S \oplus K_S^{\perp} \;\cong\; \mathbf{Z}K_S \;\oplus\; E_8(-1)$$

(cf. [DoO, VII.5]). The surface S has 240 exceptional curves (smooth rational curves E with $E^2 = -1$), cf. [DoO, p.125]. The adjunction formula shows that $EK_S = -1$ and the map $E \mapsto E + K$ gives a bijection between these exceptional curves and the roots of $E_8(-1)$, i.e. the $x \in E_8(-1)$ with $x^2 = -2$. An exceptional divisor $E \subset S$ meets the branch curve $D_0\,(\in |-2K_S|)$ of $X \to S$ in two points, hence the inverse image of E in X is a (-2)-curve. Thus we get 240 such (-2)-curves. Actually,

$$j^* : NS(S) = \mathbf{Z}K_S \;\oplus\; E_8(-1) \longrightarrow NS(X) = \mathbf{Z}L \oplus E_8(-2)$$

is the identity on the $\mathbf{Z}$-modules and $NS(X) \cong NS(S)(2)$. The class of such a (-2)-curve is $L + x$, with $x \in L^\perp = E_8(-2)$, $x^2 = -2$. As $i^*(L+x) = L - x \neq L + x$, these (-2)-curves map pairwise to conics in $\mathbf{P}^2$, which must thus be tangent to the sextic C. As also $\iota(L+x) = L - x$, these conics are invariant under $\bar{\iota}_{\mathbf{P}^2}$ and thus they correspond to plane sections of $Q \subset \mathbf{P}^3$, tangent to C_4, that is tritangent planes. This last incarnation of exceptional curves in S as tritangent planes (or equivalently, odd theta characteristics of C_4) is of course very classical.

Finally we compute the moduli. A $\bar{\iota}_{\mathbf{P}^2}$-invariant plane sextic which does not pass through $p = (1:0:0)$ has equation

$$\sum a_{ijk} x_0^{2i} x_1^j x_2^k \qquad\qquad (2i + j + k = 6,\ a_{000} \neq 0).$$

The vector space spanned by such polynomials is 16-dimensional. The subgroup of $GL(3)$ of elements commuting with $\bar{\iota}_{\mathbf{P}^2}$ (which thus preserve the eigenspaces) is isomorphic to $\mathbf{C}^* \times GL(2)$, hence the number of moduli is $16 - (1 + 4) = 11$ as expected.

Alternatively, the genus four curves whose canonical image lies on a cone have $9 - 1 = 8$ moduli (they have one vanishing even theta characteristic), next one has to specify a plane in $\mathbf{P}^3$, this gives again $8 + 3 = 11$ moduli.

3.3. **The case $\mathcal{M}_6$.** The map ϕ_L identifies X with a complete intersection of a cubic and a quadric in $\mathbf{P}^4$. According to Proposition 2.7, in suitable coordinates the Nikulin involution is induced by

$$\iota_{\mathbf{P}^4} : \mathbf{P}^4 \longrightarrow \mathbf{P}^4, \qquad (x_0 : x_1 : x_2 : x_3 : x_4) \longmapsto (-x_0 : -x_1 : x_2 : x_3 : x_4).$$

The fixed locus in $\mathbf{P}^4$ is:

$$(\mathbf{P}^4)^{\iota_{\mathbf{P}^4}} = l \cup H, \qquad l: \ x_2 = x_3 = x_4 = 0, \quad H: \ x_0 = x_1 = 0.$$

The points $X \cap l$ and $X \cap H$ are fixed points of ι on X and Proposition 2.7 shows that $\sharp(X \cap l) = 2$, $\sharp(X \cap H) = 6$. In particular, the plane H meets the quadric and cubic defining X in a conic and a cubic curve which intersect transversely. Moreover, the quadric is unique, so must be invariant under $\iota_{\mathbf{P}^4}$, and, by considering the action of $\iota_{\mathbf{P}^4}$ on the cubics in the ideal of X, we may assume that the cubic is invariant as well.

$$\begin{array}{lcl} l_{00}(x_2,x_3,x_4)x_0^2 + l_{11}(x_2,x_3,x_4)x_1^2 + l_{01}(x_2,x_3,x_4)x_0x_1 + f_3(x_2,x_3,x_4) & = & 0 \\ \alpha_{00}x_0^2 + \alpha_{11}x_1^2 + \alpha_{01}x_0x_1 + f_2(x_2,x_3,x_4) & = & 0 \end{array}$$

where the α_{ij} are constants, the l_{ij} are linear forms, and f_2, f_3 are homogeneous polynomials of degree two and three respectively. Note that the cubic contains the line $l : x_2 = x_3 = x_4 = 0$.

The projection from $\mathbf{P}^4$ to the product of the eigenspaces $\mathbf{P}^1 \times \mathbf{P}^2$ maps X to a surface defined by an equation of bidegree $(2,3)$. In fact, the equations imply that $(\sum l_{ij}x_ix_j)/f_3 = (\sum \alpha_{ij}x_ix_j)/f_2$ hence the image of X is defined by the polynomial: $(\sum l_{ij}x_ix_j)f_2 - (\sum \alpha_{ij}x_ix_j)f_3$. Adjunction shows that a smooth surface of bidegree (2,3) is a K3 surface, so the equation defines $\bar{Y}$. The space of invariant quadrics is $3+6=9$ dimensional and the space of cubics is $3\cdot 3+10=19$ dimensional. Multiplying the quadric by a linear form $a_2x_2+a_3x_3+a_4x_4$ gives an invariant cubic. The automorphisms of $\mathbf{P}^4$ commuting with ι form a subgroup which is isomorphic with $GL(2) \times GL(3)$ which has dimension $4+9=13$. So the moduli space of such K3 surfaces has dimension:

$$(9-1)+(19-1)-3-(13-1) = 11$$

as expected.

3.4. **The case $\mathcal{M}_4$.** The map $\phi_L : X \to \mathbf{P}^3$ is an embedding whose image is a smooth quartic surface. From Proposition 2.7 the Nikulin involution ι on $X \subset \mathbf{P}^3 \cong \mathbf{P}(\mathbf{C}^4)$ is induced by

$$\tilde{\iota} : \mathbf{C}^4 \longrightarrow \mathbf{C}^4, \qquad (x_0, x_1, x_2, x_3) \longmapsto (-x_0, -x_1, x_2, x_3)$$

for suitable coordinates. The eight fixed points of the involution are the points of intersection of these lines $x_0 = x_1 = 0$ and $x_2 = x_3 = 0$ with the quartic surface X.

A quartic surface which is invariant under $\tilde{\iota}$ and which does not contain the lines has an equation which is a sum of monomials $x_0^a x_1^b x_2^c x_3^d$ with $a+b = 0,2,4$ and $c+d = 4-a-b$.

The quadratic polynomials invariant under $\tilde{\iota}$ define a map:

$$\mathbf{P}^3 \longrightarrow \mathbf{P}^5, \qquad (x_0 : \ldots : x_3) \longmapsto (z_0 : z_1 : \ldots : z_5) = (x_0^2 : x_1^2 : x_2^2 : x_3^2 : x_0x_1 : x_2x_3)$$

which factors over $\mathbf{P}^3/\tilde{\iota}$. Note that any quartic invariant monomial is a monomial of degree two in the z_i. Thus if $f = 0$ is the equation of X, then $f(x_0, \ldots, x_3) = q(z_0, \ldots, z_5)$ for a quadratic

form q. This implies that

$$\bar{Y}: \qquad q(z_0,\ldots,z_5)=0, \quad z_0z_1-z_4^2=0, \quad z_2z_3-z_5^2=0$$

is the intersection of three quadrics.

The invariant quartics span a $5+9+5=19$-dimensional vector space. On this space the subgroup H of $GL(4)$ of elements which commute with $\iota_{\mathbf{P}^3}$ acts, it is easy to see that $H\cong GL(2)\times GL(2)$ (in block form). Thus $\dim H=8$ and we get an $19-8=11$ dimensional family of quartic surfaces in $\mathbf{P}^3$, as desired. See [I] for some interesting sub-families.

3.5. **The case $\mathcal{M}_{\tilde{4}}$.** In this case $\mathbf{Z}L\oplus E_8(-2)$ has index two in $NS(X)$. Choose a $v\in E_8(-2)$ with $v^2=-4$. Then we may assume that $NS(X)$ is generated by L, $E_8(-2)$ and $E_1:=(L+v)/2$, cf. (the proof of) Proposition 2.2. Let $E_2:=(L-v)/2$, then $E_i^2=L^2/4+v^2/4=1-1=0$. By Riemann-Roch we have:

$$\chi(\pm E_i)=E_i^2/2+2=2$$

and so $h^0(\pm E_i)\geq 2$ so E_i or $-E_i$ is effective. Now $L\cdot E_i=L^2/2+v/2\cdot L=2$, hence E_i is effective. As $p_a(E_i)=1$ and $E_iN\geq 0$ for all (-2)-curves N, each E_i is the class of an elliptic fibration. As $L=E_1+E_2$, by [SD, Theorem 5.2] the map ϕ_L is a 2:1 map to a quadric Q in $\mathbf{P}^3$ and it is ramified on a curve B of bi-degree $(4,4)$. The quadric is smooth, hence isomorphic to $\mathbf{P}^1\times\mathbf{P}^1$, because there are no (-2)-curves in $NS(X)$ perpendicular to L.

Let $i:X\to X$ be the covering involution of $X\to Q$. Then i and the Nikulin-involution ι commute. The elliptic pencils E_1 and E_2 are permuted by ι because $\iota^*L=L, \iota^*v=-v$. This means that the involution $\bar{\iota}_Q$ on $Q=\mathbf{P}^1\times\mathbf{P}^1$ induced by ι acts as $((s:t),(u:v))\mapsto((u:v),(s:t))$. The quotient of $Q/\bar{\iota}_Q$ is well known to be isomorphic to $\mathbf{P}^2$.

The fixed point set of $\bar{\iota}_Q$ in $\mathbf{P}^1\times\mathbf{P}^1$ is the diagonal Δ. Thus Δ intersects the branch curve B in eight points. The inverse image of these points in X are the eight fixed points of the Nikulin involution.

The diagonal maps to a conic C_0 in $\mathbf{P}^2=Q/\bar{\iota}_Q$, which gives the representation of a smooth quadric as double cover of $\mathbf{P}^2$ branched along a conic (in equations: $t^2=q(x,y,z)$). The curve B maps to a plane curve isomorphic to $\bar{B}=B/\iota$. As ι has 8 fixed points on the genus 9 curve B, the genus of $\bar{B}$ is 3 and $\bar{B}\subset\mathbf{P}^2$ is a quartic curve.

Let $j=i\iota=\iota i\in Aut(X)$. The fixed point set of j is easily seen to be the inverse image C_3 of the diagonal $\Delta\subset Q$. As $C_3\to\Delta$ branches over the 8 points in $B\cap\Delta$, C_3 is a smooth (hyperelliptic) genus three curve. Thus the surface $S:=X/j$ is smooth and the image of C_3 in S lies in the linear system $|-2K_S|$. The double cover $S\to\mathbf{P}^2$ is branched over the plane quartic $\bar{B}\subset\mathbf{P}^2$. This implies that S is a Del Pezzo surface of degree 2, cf. [Dem], [DoO].

This leads to the following diagrams of double covers and fixed point sets:

$$\begin{tikzcd} & X \arrow[dl] \arrow[d] \arrow[dr] & \\ Q \cong \mathbf{P}^1 \times \mathbf{P}^1 \arrow[dr] & \bar{Y} \arrow[d] & S \arrow[dl] \\ & \mathbf{P}^2 & \end{tikzcd} \qquad \begin{tikzcd} & B \cup C_3 \arrow[dl] \arrow[d] \arrow[dr] & \\ B \cup \Delta \arrow[dr] & \bar{B} \cup C_0 \arrow[d] & \bar{B} \cup C_3 \arrow[dl] \\ & \bar{B} \cup C_0 & \end{tikzcd}$$

In particular the eight nodal surface $\bar{Y}$ is the double cover of $\mathbf{P}^2$ branched over the reducible sextic with components the conic C_0 and the quartic $\bar{B}$. The nodes of $\bar{Y}$ map to the intersection points of C_0 and $\bar{B}$.

To count the moduli we note that the homogeneous polynomials of degree two and four in three variables span vector spaces of dimension 6 and 15, as $\dim GL(3) = 9$ we get: $(6-1) + (15-1) - (9-1) = 11$ moduli.

3.6. **The case $\mathcal{M}_8$.** We have $H^0(X, L) \cong \pi^* H^0(Y, M_1) \oplus \pi^* H^0(Y, M_2)$ and $L^2 = 8$, $M_i^2 = 2$ so $h^0(L) = 6$, $h^0(M_i) = 3$ for $i = 1, 2$. The image of X under ϕ_L is the intersection of three quadrics in $\mathbf{P}^5$ and ι is induced by

$$\tilde{\iota} : \mathbf{C}^6 \longrightarrow \mathbf{C}^6, \qquad (x_0, x_1, x_2, y_0, y_1, y_2) \longmapsto (x_0, x_1, x_2, -y_0, -y_1, -y_2).$$

The multiplication map maps the 21-dimensional space $S^2H^0(X, L)$ onto the 18-dimensional space $H^0(X, 2L)$. Using ι we can get some more information on the kernel of this map, which are the quadrics defining $X \subset \mathbf{P}^5$. We have:

$$S^2H^0(X, L) \cong \left(S^2H^0(Y, M_1) \oplus S^2H^0(Y, M_2)\right) \oplus \left(H^0(Y, M_1) \otimes H^0(Y, M_2)\right),$$

Moreover, as

$$\beta^*(2L) = \pi^* M, \qquad \text{with} \quad M = 2M_1 + N_1 + \ldots + N_4 = 2M_2 + N_5 + \ldots + N_8,$$

(cf. Proposition 2.7) we have the decomposition

$$H^0(X, 2L) \cong \pi^* H^0(Y, M) \oplus \pi^* H^0(Y, M - \hat{N}), \qquad (h^0(M) = (M^2)/2 + 2 = 10,\ h^0(M - \hat{N}) = 8).$$

In particular, the multiplication maps splits as:

$$H^0(M_1) \otimes H^0(M_2) \longrightarrow H^0(Y, M - \hat{N})$$

(vector spaces with dimensions with $3 \cdot 3 = 9$ and 8 resp.) and

$$S^2H^0(Y, M_1) \ \oplus\ S^2H^0(Y, M_2) \longrightarrow H^0(Y, M)$$

(with dimensions $6 + 6 = 12$ and 10 resp.). Each of these two maps is surjective, and as $S^2H^0(Y, M_1) \to H^0(Y, M)$ is injective (ϕ_{M_1} maps Y onto $\mathbf{P}^2$), the quadrics in the ideal of X can be written as:

$$Q_1(x) - Q_2(y) = 0, \qquad Q_3(x) - Q_4(y) = 0, \qquad B(x, y) = 0$$

with Q_i homogeneous of degree two in three variables, and B of bidegree $(1,1)$. Note that each eigenspace intersects X in $2 \cdot 2 = 4$ points.

The surface $\bar{Y}$ maps to $\mathbf{P}^2 \times \mathbf{P}^2$ with the map $\phi_{M_1} \times \phi_{M_2}$, its image is the image of X under the projections to the eigenspaces $\mathbf{P}^5 \to \mathbf{P}^2 \times \mathbf{P}^2$. As $(x_0 : \ldots : y_2) \mapsto Q_1(x)/Q_2(y)$ is a constant rational function on X and similarly for $Q_3(x)/Q_4(y)$, there is a $c \in \mathbf{C}$ such that the image of X is contained in the complete intersection of type $(2,2)$, $(1,1)$ in $\mathbf{P}^2 \times \mathbf{P}^2$ defined by

$$Q_1(x)Q_4(y) - cQ_3(x)Q_2(y) = 0, \qquad B(x,y) = 0.$$

By adjunction, smooth complete intersections of this type are K3 surfaces.

To count the moduli, note that the first two equations come from a $6+6=12$-dimensional vector space and the third comes from a $3 \cdot 3 = 9$-dimensional space. The Grassmanian of 2-dimensional subspaces of a 12 dimensional space has dimension $2(12-2) = 20$. The subgroup of $GL(6)$ which commutes with $\iota_{\mathbf{P}^5}$ is isomorphic to $GL(3) \times GL(3)$ and has dimension $9+9=18$. Thus we get $20 + (9-1) - (18-1) = 11$ moduli, as expected.

3.7. **The case $\mathcal{M}_{\tilde{8}}$.** We have $H^0(X,L) \cong \pi^*H^0(Y,M) \oplus \pi^*H^0(Y,M-\hat{N})$ and $L^2 = 8$, $M^2 = 4$ so $h^0(M) = 4, h^0(M-N) = 2$. The image of X under ϕ_L is is the intersection of three quadrics in $\mathbf{P}^5$ and ι is induced by

$$\tilde{\iota} : \mathbf{C}^6 \longrightarrow \mathbf{C}^6, \qquad (x_0,x_1,x_2,x_3,y_0,y_1) \longmapsto (x_0,x_1,x_2,x_3,-y_0,-y_1).$$

To study the quadrics defining X, that is the kernel of the multiplication map $S^2H^0(X,L) \to H^0(X,2L)$ we again split these spaces into ι^*-eigenspaces:

$$S^2H^0(X,L) \cong \left(S^2H^0(Y,M) \oplus S^2H^0(Y,M-\hat{N})\right) \oplus \left(H^0(Y,M) \otimes H^0(Y,M-\hat{N})\right),$$

(with dimensions $21 = (10+3)+8$) and

$$H^0(X,2L) \cong \pi^*H^0(Y,2M) \oplus \pi^*H^0(Y,2M-\hat{N})$$

(with dimensions $h^0(2M) = 10,\ h^0(2M-\hat{N}) = 8$).

This implies that there are no quadratic relations in the 8 dimensional space $H^0(Y,M) \otimes H^0(Y,M-\hat{N})$. As ϕ_M maps Y onto a quartic surface in $\mathbf{P}^3$ and $M-\hat{N}$ is a map of Y onto $\mathbf{P}^1$, the quadrics in the ideal of X are of the form:

$$y_0^2 = Q_1(x), \qquad y_0y_1 = Q_2(x), \qquad y_1^2 = Q_3(x).$$

The fixed points of the involution are the eight points in the intersection of X with the $\mathbf{P}^3$ defined by $y_0 = y_1 = 0$.

The image of Y by ϕ_M is the image of the projection of X from the invariant line to the invariant $\mathbf{P}^3$, which is defined by $y_0 = y_1 = 0$. The image is the quartic surface defined by $Q_1Q_3 - Q_2^2 = 0$ which can be identified with $\bar{Y}$. The equation is the determinant of a symmetric 2×2 matrix, which also implies that this surface has 8 nodes, (cf. [Ca, Theorem 2.2], [B, section 3]), the nodes form an even set (cf. [Ca, Proposition 2.6]).

We compute the number of moduli. Quadrics of this type span a space U of dimension $3+10 = 13$. The dimension of the Grassmanian of three dimensional subspaces of U is $3(13-3) = 30$. The group of automorphisms of $\mathbf{C}^6$ which commute with $\iota_{\mathbf{P}^5}$ is $GL(2) \times GL(4)$. So we have a $30 - (4+16-1) = 11$ dimensional space of such K3-surfaces in $\mathbf{P}^5$, as expected.

3.8. **The case $\mathcal{M}_{12}$.** We have $H^0(X,L) \cong \pi^* H^0(Y,M_1) \oplus \pi^* H^0(Y,M_2)$ and $L^2 = 12$, $M_i^2 = 4$ so $h^0(L) = 8$, $h^0(M_i) = 4$ for $i = 1,2$. The image of X under ϕ_L is the intersection of ten quadrics in $\mathbf{P}^7$.

Following Example 3.6, we use ι^* to split the multiplication map from the $36 = (10+10)+16$-dimensional space $S^2H^0(X,L)$ onto the $26 = 14+12$-dimensional space $H^0(X,2L)$, again $\beta^*(2L) = \pi^* M$ for an $M \in NS(Y)$ with $M^2 = 24$. Thus we find $20 - 14 = 6$ quadrics of the type $Q_1(x) - Q_2(y)$ with Q_i quadratic forms in 4 variables, and $16 - 12 = 4$ quadratic forms $B_i(x,y)$, $i = 1, \dots, 4$ where x, y are coordinates on the two eigenspaces in $H^0(X,L)$.

In particular, the projection from $\mathbf{P}^7$ to the product of the eigenspaces $\mathbf{P}^3 \times \mathbf{P}^3$ maps X onto a surface defined by 4 equations of bidegree $(1,1)$. Adjunction shows that a complete intersection of this type is a K3 surface, so the four B_i's define $\bar{Y} \subset \mathbf{P}^3 \times \mathbf{P}^3$.

Each B_i can be written as: $B_i(x,y) = \sum_j l_{ij}(x) y_j$ with linear forms l_{ij} in $x = (x_0, \dots, x_3)$. The image of $\bar{Y} \subset \mathbf{P}^3 \times \mathbf{P}^3$ under the projection to the first factor is then defined by $\det(l_{ij}(x)) = 0$, which is a quartic surface in $\mathbf{P}^3$ as expected. In fact, a point $x \in \mathbf{P}^3$ has a non-trivial counter image $(x,y) \in X \subset \mathbf{P}^3 \times \mathbf{P}^3$ iff the matrix equation $(l_{ij})y = 0$ has a non-trivial solution.

As X is not a complete intersection, we omit the moduli count.

3.9. **The case $\mathcal{M}_{\widetilde{12}}$.** In this case $\beta^* L \cong \pi^* M$, $h^0(L) = 8 = 5 + 3 = h^0(M) + h^0(M - \hat{N})$. We consider again the quadrics in the ideal of X in Example 3.7. The space $S^2H^0(X,L)$ of quadrics on $\mathbf{P}^7$ decomposes as:

$$S^2H^0(X,L) \cong \left(S^2H^0(Y,M) + S^2H^0(Y,M-\hat{N}) \right) \oplus \left(H^0(Y,M) \otimes H^0(Y,M-\hat{N}) \right),$$

with dimensions $36 = (15+6)+15$, whereas the sections of $2L$ decompose as:

$$h^0(2L) = (4L^2)/2 + 2 = 26 = 14 + 12 = h^0(2M) \oplus h^0(2M - \hat{N}).$$

Thus there are $(15+6) - 14 = 7$ independent quadrics in the ideal of $X \subset \mathbf{P}^7$ which are invariant and there are $15 - 12 = 3$ quadrics which are anti-invariant under the map

$$\tilde{\iota} : \mathbf{C}^8 \longrightarrow \mathbf{C}^8, \qquad (x_0, \dots, x_4, y_0, \dots, y_2) \longmapsto (x_0, \dots, x_4, -y_0, \dots, -y_2).$$

An invariant quadratic polynomial looks like $q_0(x_0, \dots, x_4) + q_1(y_0, y_1, y_2)$, and since the space of quadrics in three variables is only 6 dimensional, there is one non-zero quadric q in the ideal of the form $q = q(x_0, \dots, x_4)$. An anti-invariant quadratic polynomial is of bidegree $(1,1)$ in x and y. In particular, the image of the projection of X to the product of the eigenspaces $\mathbf{P}^4 \times \mathbf{P}^2$ is contained in one hypersurface of bidegree $(2,0)$ and in three hypersurfaces of bidegree $(1,1)$. The complete intersection of four general such hypersurfaces is a K3 surface (use adjunction and $(2 + 3 \cdot 1, 3 \cdot 1) = (5,3)$).

The three anti-invariant quadratic forms can be written as $\sum_j l_{ij}(x) y_j$, $i = 1,2,3$. The determinant of the 3×3 matrix of linear forms $(l_{ij}(x))$, defines a cubic form which is an equation for the image of X in $\mathbf{P}^4$ (cf. Example 3.8). Thus the projection $\bar{Y}$ of X to $\mathbf{P}^4$ is the intersection of the quadric defined by $q(x) = 0$ and a cubic.

The projection to $\mathbf{P}^2$ is 2:1, as it should be, since for general $y \in \mathbf{P}^2$ the three linear forms in x given by $\sum_j l_{ij}(x) y_j$ define a line in $\mathbf{P}^4$ which cuts the quadric $q(x) = 0$ in two points.

4. Elliptic fibrations with a section of order two

4.1. **Elliptic fibrations and Nikulin involutions.** Let X be a K3 surface which has an elliptic fibration $f : X \to \mathbf{P}^1$ with a section σ. The set of sections of f is a group, the Mordell-Weil group MW_f, with identity element σ. This group acts on X by translations and these translations preserve the holomorphic two form on X. In particular, if there is an element $\tau \in MW_f$ of order two, then translation by τ defines a Nikulin involution ι.

In that case the Weierstrass equation of X can be put in the form:

$$X : \qquad y^2 = x(x^2 + a(t)x + b(t))$$

the sections σ, τ are given by the section at infinity and $\tau(t) = (x(t), y(t)) = (0, 0)$. For the general fibration on a K3 surface X, the degrees of a and b are 4 and 8 respectively.

4.2. **Proposition.** Let $X \to \mathbf{P}^1$ be a general elliptic fibration with sections σ, τ as above in section 4.1. and let ι be the corresponding Nikulin involution on X. These fibrations form a 10-dimensional family.

The quotient K3 surface Y also has an elliptic fibration:

$$Y : \qquad y^2 = x(x^2 - 2a(t)x + (a(t)^2 - 4b(t)),$$

We have:

$$NS(X) \cong NS(Y) \cong U \oplus N, \qquad T_X \cong T_Y \cong U^2 \oplus N.$$

The bad fibers of $X \to \mathbf{P}^1$ are eight fibers of type I_1 (which are rational curves wit a node) over the zeroes of $a^2 - 4b$ and eight fibers of type I_2 (these fibers are the union of two $\mathbf{P}^1$'s meeting in two points) over the zeroes of b. The bad fibers of $Y \to \mathbf{P}^1$ are eight fibers of type I_2 over the zeroes of $a^2 - 4b$ and eight fibers of type I_1 over the zeroes of b.

Proof. Since X has an elliptic fibration with a section, $NS(X)$ contains a copy of the hyperbolic plane U (with standard basis the class of a fiber f and $f + \sigma$). The discriminant of the Weierstrass model of X is $\Delta_X = b^2(a^2 - 4b)$ and the fibers of the Weierstrass model over the zeroes of Δ_X are nodal curves. Thus $f : X \to \mathbf{P}^1$ has eight fibers of type I_1 (which are rational curves with a node) over the zeroes of $a^2 - 4b$ and 8 fibers of type I_2 (these fibers are the union of two $\mathbf{P}^1$'s meeting in two points) over the zeroes of b.

The components of the singular fibers which do not meet the zero section σ, give a sublattice $< -2 >^8$ perpendicular to U. If there are no sections of infinite order, the lattice $U \oplus < -2 >^8$ has finite index in the Néron Severi group of X. Hence X has $22 - 2 - 10 = 10$ moduli. One can also appeal to [Shim] where the Néron Severi group of the general elliptic K3 fibration with a section of order two is determined. To find the moduli from the Weierstrass model, note that a and b depend on $5 + 9 = 14$ parameters. Using transformations of the type $(x, y) \mapsto (\lambda^2 x, \lambda^3 y)$ (and dividing the equation by λ^6) and the automorphism group $PPG2$) of $\mathbf{P}^1$ we get $14 - 1 - 3 = 10$ moduli.

The Shioda-Tate formula (cf. e.g. [Shio, Corollary 1.7]) shows that the discriminant of the Néron Severi group is $2^8/n^2$ where n is the order of the torsion subgroup of MW_f. The curve defined by $x^2 + a(t)x + b(t) = 0$ cuts out the remaining pair of points of order two on each smooth fiber. As it is irreducible in general, MW_f must be cyclic. If there were a section σ of

order four, it would have to satisfy $2\sigma = \tau$. But in a fiber of type I_2 the complement of the singular points is the group $G = \mathbf{C}^* \times (\mathbf{Z}/2\mathbf{Z})$ and the specialization $MW_f \to G$ is an injective homomorphism. Now τ specializes to $(\pm 1, \bar{1})$ (the sign doesn't matter) since τ specializes to the node in the Weierstrass model. But there is no $g \in G$ with $2g = (\pm 1, \bar{1})$. We conclude that for general X we have $MW_f = \{\sigma, \tau\} \cong \mathbf{Z}/2\mathbf{Z}$ and that the discriminant of the Néron Severi group of X is 2^6.

The Néron Severi group has $\mathbf{Q}$ basis $\sigma, f, N_1, \ldots, N_8$ where the N_i are the components of the I_2 fibers not meeting σ. As $\tau \cdot \sigma = 0$, $\tau \cdot f = 1$ and $\tau \cdot N_i = 1$, we get:

$$\tau = \sigma + 2f - \hat{N}, \qquad \hat{N} = (N_1 + \ldots + N_8)/2.$$

Thus the smallest primitive sublattice containing the N_i is the Nikulin lattice. Comparing discriminants we conclude that:

$$NS(X) = \langle s, f\rangle \oplus \langle N_1, \ldots, N_8, \hat{N}\rangle \;\cong\; U \oplus N.$$

The transcendental lattice T_X of X can be determined as follows. It is a lattice of signature $(2+, 10-)$ and its discriminant form is the opposite of the one of N, but note that $q_N = -q_N$ since q_N takes values in $\mathbf{Z}/2\mathbf{Z}$. Moreover, $T_X^*/T_X \cong N^*/N \cong (\mathbf{Z}/2\mathbf{Z})^6$. Using [Ni2, Corollary 1.13.3], we find that T_X is uniquely determined by the signature and the discriminant form. The lattice $U^2 \oplus N$ has these invariants, so

$$T_X \cong U^2 \oplus N.$$

As the Nikulin involution preserves the fibers of the elliptic fibration on X, the desingularisation Y of the quotient X/ι has an elliptic fibration $g : Y \to \mathbf{P}^1$, with a section $\bar{\sigma}$, (the image of σ). The Weierstrass equation of Y can be found from [ST, p.79].

The discriminant of the Weierstrass model of Y is $\Delta_Y = 4b(a^2 - 4b)^2$ and, reasoning as before, we find the bad fibers of $g : Y \to \mathbf{P}^1$. In particular, the I_1 and I_2 fibers of X and Y are indeed 'interchanged'.

Geometrically, the reason for this is as follows. The fixed points of translation by τ are the eight nodes in the I_1-fibers, blowing them up gives I_2-type fibers which map to I_2-type fibers in Y. The exceptional curves lie in the ramification locus of the quotient map, the other components, which meet σ, map 2:1 to components of the I_2-fibers which meet $\bar{\sigma}$. The two components of an I_2-fiber in X are interchanged and also the two singular points of the fiber are permuted, so in the quotient this gives an I_1-type fiber. □

4.3. **Remark.** Note that $NS(X) \oplus T_X \cong U^3 \oplus N^2$, however, there is *no* embedding of N^2 into $E_8(-1)^2$, such that $N \oplus \{0\}$ $(\subset NS(X))$ is primitive in $E_8(-1)^2$. However, $N^2 \subset \Gamma_{16}(-1)$ (cf. section 1.11), an even, negative definite, unimodular lattice of rank 16 and $U^3 \oplus \Gamma_{16}(-1) \cong U^3 \oplus E_8(-1)^2$ by the classification of even indefinite unimodular quadratic forms.

4.4. **Morrison-Nikulin involutions.** D. Morrison observed that a K3 surface X having two perpendicular copies of $E_8(-1)$ in the Néron Severi group has a Nikulin involution which exchanges the two copies of $E_8(-1)$, cf. [Mo, Theorem 5.7]. We will call such an involution a Morrison-Nikulin involution. This involution then has the further property that $T_Y \cong T_X(2)$

where Y is the quotient K3 surface and we have a Shioda-Inose structure on Y (cf. [Mo, Theorem 6.3])

4.5. **Moduli.** As $E_8(-1)$ has rank eight and is negative definite, a projective K3 surface with a Morrison-Nikulin involution has a Néron Severi group of rank at least 17 and hence has at most three moduli. In case the Néron Severi group has rank exactly 17, we get

$$NS(X) \cong \langle 2n \rangle \oplus E_8(-1) \oplus E_8(-1)$$

since the sublattice $E_8(-1)^2$ is unimodular. Results of Kneser and Nikulin, [Ni2, Corollary 1.13.3], guarantee that the transcendental lattice $T_X := NS(X)^\perp$ is uniquely determined by its signature and discriminant form. As the discriminant form of T_X is the opposite of the one on $NS(X)$ we get

$$T_X \cong \langle -2n \rangle \oplus U^2.$$

In case $n = 1$ such a three dimensional family can be obtained from the double covers of $\mathbf{P}^2$ branched along a sextic curve with two singularities which are locally isomorphic to $y^3 = x^5$. The double cover then has two singular points of type E_8, that is, each of these can be resolved by eight rational curves with incidence graph E_8. As the explicit computations are somewhat lengthy and involved, we omit the details. See [P] and [Deg] for more on double covers of $\mathbf{P}^2$ along singular sextics.

4.6. **Morrison-Nikulin involutions on elliptic fibrations.** We consider a family of K3 surfaces with an elliptic fibration with a Morrison-Nikulin involution induced by translation by a section of order two. It corresponds to the family with $n = 2$ from section 4.5.

Note that in the proposition below we describe a $K3$ surface Y with a Nikulin involution and quotient K3 surface X such that $T_Y = T_X(2)$, which is the 'opposite' of what would happen if the involution of Y was a Morrison-Nikulin involution. it is not hard to see that there is no primitive embedding $T_Y \hookrightarrow U^3$, so Y does not have a Morrison-Nikulin involution at all (cf. [Mo, Theorem 6.3]).

4.7. **Proposition.** Let $X \to \mathbf{P}^1$ be a general elliptic fibration defined by the Weierstrass equation

$$X: \qquad y^2 = x(x^2 + a(t)x + 1), \qquad a(t) = a_0 + a_1 t + a_2 t^2 + t^4 \in \mathbf{C}[t].$$

The K3 surface X has a Morrison-Nikulin involution defined by translation by the section, of order two, $t \mapsto (x(t), y(t)) = (0, 0)$. Then:

$$NS(X) = \langle 4 \rangle \oplus E_8(-1) \oplus E_8(-1), \qquad T_X = \langle -4 \rangle \oplus U^2.$$

The bad fibers of the fibration are nodal cubics (type I_1) over the eight zeroes of $a^2(t) - 4$ and one fiber of type I_{16} over $t = \infty$.

The quotient K3 surface Y has an elliptic fibration defined by the Weierstrass model:

$$Y: \qquad y^2 = x(x^2 - 2a(t)x + (a(t)^2 - 4)), \qquad T_Y \cong \langle -8 \rangle \oplus U(2)^2.$$

This K3 surface has a Nikulin involution defined by translation by the section $t \mapsto (x(t), y(t)) = (0, 0)$ and the quotient surface is X. For general X, the bad fibers of Y are 8 fibers of type I_2

over the same points in $\mathbf{P}^1$ where X has fibers of type I_1 and at infinity Y has a fiber of type I_8.

Proof. As we observed in section 4.1, translation by the section of order two defines a Nikulin involution.

Let $\hat{a}(s) := s^4 a(s^{-1})$, it is a polynomial of degree at most four and $\hat{a}(0) \neq 0$. Then on $\mathbf{P}^1 - \{0\}$, with coordinate $s = t^{-1}$, the Weierstrass model is

$$v^2 = u(u^2 + \hat{a}(s)u + s^8), \qquad \Delta = s^{16}(\hat{a}(s)^2 - 4s^8), \qquad u = s^4 x,\ v = s^6 y,$$

where Δ is the discriminant. The fiber over $s = 0$ is a stable (nodal) curve, so the corresponding fiber X_∞ is of type I_m where m is the order of vanishing of the discriminant in $s = 0$ (equivalently, it is the order of the pole of the j-invariant in $s = 0$). Thus X_∞ is an I_{16} fiber. As the section of order two specializes to the singular point $(u, v, s) = (0, 0, 0)$, after blow up it will not meet the component of the fiber which meets the zero section.

The group structure of the elliptic fibration induces a Lie group structure on the smooth part of the I_{16} fiber. Taking out the 16 singular points in this fiber, we get the group $\mathbf{C}^* \times \mathbf{Z}/16\mathbf{Z}$. The zero section meets the component C_0, where

$$C_n \ := \ \mathbf{P}^1 \times \{\bar{n}\} \ \hookrightarrow X_\infty,$$

and the section of order two must meet C_8. Translation by the section of order two induces the permutation $C_n \ \mapsto \ C_{n+8}$ of the 16 components of the fiber. The classes of the components C_n, with $n = -2, \ldots, 4$, generate a lattice of type $A_7(-1)$ which together with the zero section gives an $E_8(-1)$. The Nikulin involution maps this $E_8(-1)$ to the one whose components are the C_n, $n = 6, \ldots, 12$, and the section of order two. Thus the Nikulin involution permutes two perpendicular copies of $E_8(-1)$ and hence it is a Morrison-Nikulin involution.

The bad fibers over $\mathbf{P}^1 - \{\infty\}$ correspond to the zeroes of $\Delta = a^2(t) - 4$. For general a, Δ has eight simple zeroes and the fibers are nodal, so we have eight fibers of type I_1 in $\mathbf{P}^1 - \{\infty\}$.

By considering the points on $\mathbf{P}^1$ where there are bad fibers it is not hard to see that we do get a three dimensional family of elliptic K3 surfaces with a Morrison-Nikulin involution. Thus the general member of this three dimensional family has a Néron Severi group S of rank 17.

As we constructed a unimodular sublattice $E_8(-1)^2 \subset S$, we get $S \cong < -d > \oplus E_8(-1)^2$ and d (> 0) is the discriminant of S. The Shioda-Tate formula (cf. e.g. [Shio, Corollary 1.7]) gives that $d = 16/n^2$ where n is the order of the group of torsion sections. As n is a multiple of 2 and d must be even it follows that $d = 4$. As the embedding of $NS(X)$ into $U^3 \oplus E_8(-1)^2$ is unique up to isometry it is easy to determine $T_X = NS(X)^\perp$. Finally $T_Y \cong T_X(2)$ by the results of [Mo].

The Weierstrass model of the quotient elliptic fibration Y can be computed with the standard formula cf. [ST, p.79], the bad fibers can be found from the discriminant $\Delta = -4(a^2 - 4)^2$ (and j-invariant). Alternatively, fixed points of the involution on X are the nodes in the I_1-fibers. Since these are blown up, we get 8 fibers of type I_2 over the same points in $\mathbf{P}^1$ where X has fibers of type I_1. At infinity Y has a fiber of type I_8 because the involution on X permutes of the 16 components of the I_{16}-fiber ($C_n \leftrightarrow C_{n+8}$). $\square$

4.8. **Remark.** The Weierstrass model we used to define X, $y^2 = x(x^2 + a(t)x + 1)$, exhibits X as the minimal model of the double cover of $\mathbf{P}^1 \times \mathbf{P}^1$, with affine coordinates x and t. The branch curve consists of the the lines $x = 0$, $x = \infty$ and the curve of bidegree $(2, 4)$ defined by $x^2 + a(t)x + 1 = 0$. Special examples of such double covers are studied in section V.23 of [BPV]. In particular, on p.185 the 16-gon appears with the two sections attached and the E_8's are pointed out in the text. Note however that our involution is *not* among those studied there.

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